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NONLINEAR CURVATURE EXPRESSIONS FOR COMBINED FLAPWISE BENDING, CHORDWISE BENDING, TORSION AND EXTENSION OF TWISTED ROTOR BLADES

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16. Abstract

The nonlinear curvature expressions for a twisted rotor blade or a beam undergoing transverse bending in two planes, torsion, and extension are developed. The curvature expressions are obtained using simple geometric considerations, in contrast to other methods described in the literature. The expressions are first developed in a general manner using the geometrical nonlinear theory of elasticity. These general nonlinear expressions are then systematically reduced to four levels of approximation by imposing various simplifying assumptions, and in each of these levels the second-degree nonlinear expressions are given. The assumptions are carefully stated and their implications with respect to the nonlinear theory of elasticity as applied to beams are pointed out. The transformation matrices between the deformed and undeformed blade-fixed coordinates, which are needed in the development of the curvature expressions, are also given for three of the levels of approximation. The present curvature expressions and transformation matrices are compared with corresponding expressions existing in the literature. These comparisons indicate some discrepancies with the present results in the nonlinear terms. The reasons for these discrepancies are explained. Since both the nonlinear curvature expressions and the nonlinear transformation matrices are needed to develop the nonlinear aeroelastic equations of motion of flexible rotor blades, the effect on stability of the discrepancies has yet to be assessed. As a by-product of this study the controversy regarding whether the uncoupled extensional frequency of a rotating beam increases or decreases with increasing rotational speed is resolved.

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SUMMARY

The nonlinear curvature expressions for a twisted rotor blade or a beam undergoing transverse bending in two planes, torsion, and extension are developed. The curvature expressions are obtained using simple geometric considerations, in contrast to other methods described in the literature. The expressions are first developed in a general manner using the geometrical nonlinear theory of elasticity. These general nonlinear expressions are then systematically reduced to four levels of approximation by imposing various simplifying assumptions, and in each of these levels the second-degree nonlinear expressions are given. The assumptions are carefully stated and their implications with respect to the nonlinear theory of elasticity as applied to beams are pointed out. The transformation matrices between the deformed and undeformed blade-fixed coordinates, which are needed in the development of the curvature expressions, are also given for three of the levels of approximation. The present curvature expressions and transformation matrices are compared with corresponding expressions existing in the literature. These comparisons indicate some discrepancies with the present results in the

nonlinear terms. The reasons for these discrepancies are explained. Since both the nonlinear curvature expressions and the nonlinear transformation matrices are needed to develop the nonlinear aeroelastic equations of motion of flexible rotor blades, the effect on stability of the discrepancies has yet to be assessed. As a by-product of this study the controversy regarding whether the uncoupled extensional frequency of a rotating beam increases or decreases with increasing rotational speed is resolved.

INTRODUCTION

Aeroelastic stability associated with flap-lag-torsion-axial coupling of flexible helicopter rotor blades involves both linear and nonlinear coupling among the degrees of freedom. If the second-degree nonlinearities have to be considered, the governing equations of motion must include all the second degree nonlinear terms. To develop these nonlinear equations a need thus arises both for the second-degree nonlinear expressions for the bending curvatures and torsional curvature (sometimes called "torsion" or "total twist") and the second-degree nonlinear transformation matrix between the deformed and undeformed blade-fixed coordinates. The curvatures are also needed in nonlinear static and dynamic analyses of tubes transporting fluids, propellers, etc. Love (ref. 1), using Kirchhoff's kinetic analogy, in conjunction with direction cosines, developed the linear curvature expressions for an inextensible curved and twisted beam. Hunter (ref 2), using a vectorial approach, extended Love's work to the nonlinear case without invoking the inextensibility assumption. Another vector extension of Love's work to the nonlinear case was given by Doll and Mote (ref. 3). References 1 to 3 all treat the problem which includes initial bending curvatures and initial twist. A special case of this problem which is receiving considerable attention in the literature is that of a flexible hingeless rotor blade having only initial twist (sometimes called "pretwist" or "built-in twist"). Hodges (ref. 4) developed the nonlinear curvature expressions for a blade with zero pretwist following the approach given by Novozhilov (ref. 5). Nonlinear curvature expressions for an elastic blade were also given in references 6 and 7 where they were obtained by solving a differential equation for the transformation matrix relating the blade-fixed coordinates of the deformed and undeformed blade. The nonlinear curvatures were also given in reference 8 where they were obtained from simple geometric considerations in combination with Kirchhoff's kinetic analogy.

In this paper the approach employed in reference 8 will be extended to obtain the general nonlinear curvature expressions using the nonlinear theory of elasticity. In general, the nonlinearity of the equations of the theory of

elasticity can have both geometrical and physical origin. Geometric nonlinearity is associated with the necessity to consider the deformed configuration to write the equilibrium equations and the need to include nonlinear terms in the strain-displacement relations. Physical nonlinearity is associated with the necessity to consider the relations between the components of stress and strain as nonlinear. In the present development only geometrical nonlinearity is considered. The general large deformation expressions for the curvatures developed herein are systematically reduced to four levels of approximation by imposing various simplifying assumptions (figure 1). The levels of approximation addressed are: (1) the particular case of large deformations in which the elongations and shears are less than unity with no restrictions on the rotations; (2) the first case of small deformations in which the elongations and shears are negligible compared to unity with no restrictions on the rotations; (3) the second case of small deformations in which the elongations, shears, and rotations are negligible compared to unity; (4) the classical linear case of small deformations in which the elongations, shears, and rotations are negligible compared to unity and the squares and products of the rotations are neglected compared to the strains. Here, and in the subsequent discussions, the terms "elongations", "shears", and "rotations" have the same meaning as in reference 5. The curvature expressions and the transformation matrices will be compared with corresponding expressions existing in the literature wherever possible.

It should be remarked that, for convenience, the case of small deformations I is obtained as a special case of the general case of large deformations rather than from the particular case of large deformations addressed herein. This consideration is also reflected in the form of the block diagram given in figure 1.

In deriving curvature expressions for a deformed blade or beam the need arises to employ Eulerian-type angles to effect a transformation between deformed and undeformed blade-fixed coordinates. If nonlinear curvatures are being developed these angles must be treated as finite rotations. Since transformation matrices corresponding to finite angles of rotation are not commutative, the order in which the rotations are imposed is important. A preliminary investigation of the nonlinear curvature expressions as influenced

by the order in which the rotational transformations between the deformed and undeformed blade-fixed coordinates are imposed was also given in reference 8. Of the six rotational transformation sequences possible reference 8 considered two: flap-lag-pitch and lag-flap-pitch. It was shown there that the torsional curvature expression for an assumed flap-lag-pitch rotational sequence differed from the torsional curvature expression for a lag-flap-pitch rotational sequence. The present paper also examines more completely the effect of these two rotational transformation sequences on the curvature expressions.

SYMBOLS

ℓ_i, m_i, n_i	direction cosines ($i = 1, 2, 3$)
s_3	coordinate along deformed elastic axis
u, v, w	elastic deformations of arbitrary point on elastic axis in radial, edgewise, and flapwise directions, respectively
XYZ	inertial axes in hub plane with origin at hub center-line
xyz	blade-fixed axis system which translates with respect to $x_o y_o z_o$
$x_o y_o z_o$	blade-fixed axis system, after deformation, which translates with respect to $x_o y_o z_o$
$x_3 y_3 z_3$	blade-fixed orthogonal axis system in deformed configuration obtained by rotating xyz; x_3-axis is tangent to the deformed elastic axis
β, ζ, θ	Eulerian-type rotational angles between xyz and $x_3 y_3 z_3$
θ_{pt}	built-in twist angle (initial twist; pretwist)
ε	extensional component of Green's strain tensor
ϕ	twist about deformed elastic axis
$\omega_{x_3}, \omega_{y_3}, \omega_{z_3}$	torsional curvature (total rotation rate about x_3 axis) and bending curvatures, respectively

Special notation:

$(\ )^{+}$	derivative with respect to s_3, $d(\ )/ds_3$
$(\ )'$	derivative with respect to x, $d(\ )/dx$

ANALYTICAL DEVELOPMENT

In this section the general nonlinear curvature expressions and the associated transformation matrices between the deformed and undeformed blade coordinates for a helicopter rotor blade will first be derived for the case in which there are no restrictions on the elongations, shears, and rotations. Simple geometric considerations will be used. The general expressions will then be systematically reduced to four levels of approximation. In each of these levels, the second-degree curvature expressions will be given. For the three cases of small deformation considered, the transformation matrices will also be given. Finally, the geometric development will be supplemented by a complementary derivation based on a direction cosine approach.

Geometric Approach

General case of large deformations. - A schematic representation of the undeformed and deformed blade geometries associated with both a flap-lag-pitch and a lag-flap-pitch rotational transformation sequence is shown in figures 2 and 3, respectively. The translational elastic deformations experienced by an arbitrary point on the elastic axis of the blade are denoted by u, v, w. The coordinate axes XYZ are inertial axes situated at the root end of the (nonrotating) blade; $x_o y_o z_o$ are axes fixed to the blade at an arbitrary point on the elastic axis of the undeformed blade. Before deformation $x_o y_o z_o$ are parallel to XYZ. Deformations u, v, w, and ϕ displace the $x_o y_o z_o$ triad to xyz and rotate xyz to $x_3 y_3 z_3$ where the axis x_3 is tangent to the deformed elastic axis. The rotation of the triad xyz to its final position $x_3 y_3 z_3$ may be expressed in terms of the Eulerian-type angles* β, ζ, and θ as shown in figure 4 for a flap-lag-pitch rotational transformation sequence and in figure 5 for a lag-flap-pitch rotational transformation sequence. These two figures reflect the use of a Lagrangian description to describe the deformed geometry of a blade element. Consistent with this description, Green's strain

*Angles of this type are further discussed in references 8 and 9.

tensor will be employed to define the strains in terms of the deformations. References 2, 8, and 9 showed that the nonlinear curvature expressions are dependent on the order in which the rotational transformations between the deformed and undeformed blade-fixed coordinates are imposed. In the present development the bending and torsional curvatures corresponding to the two rotational transformation sequences, flap-lag-pitch and lag-flap-pitch, are derived for the general case of large deformations using simple geometric considerations and then specialized to four levels of approximation.

Flap-lag-pitch transformation sequence: For this rotational transformation sequence (figure 4) the rotations are imposed as follows:

1. A positive rotation β about the negative y axis resulting in $x_1y_1z_1$.
2. A positive rotation ζ about the z_1 axis resulting in $x_2y_2z_2$
3. A positive rotation θ about the x_2 axis resulting in $x_3y_3z_3$

Figure 4 reflects the use of Kirchhoff's kinetic analogy. This analogy (see, e.g., refs. 1, 8, and 9) states that if the origin of the $x_3y_3z_3$ coordinates moves along the deflected and twisted elastic axis with a unit linear velocity and the y_3 and z_3 axes occupy, at each instant, positions corresponding to the normal and binormal directions of the deformed elastic axis*, then the angular velocities of the $x_3y_3z_3$ system with respect to the xyz system are expressed in the same way as the corresponding bending and torsional curvatures. One then merely has to replace the time derivative in the expressions for the angular velocities by a space derivative with respect to the (curvilinear) coordinate s_3 along the elastic axis of the deformed blade. The space derivatives of the rotation angles with respect to s_3, β^+, ζ^+, and θ^+, as shown in figure 4, assume the role of the angular velocities in application of the analogy. The bending curvatures ω_{y3} and ω_{z3}, and the torsional curvature ω_{x3}, are obtained by projecting β^+, ζ^+, and θ^+ along the $x_3y_3z_3$ axes and

*For the general case of large deformations, the principal axes of the deformed blade do not coincide with the normal and binormal directions of the deformed elastic axis (see refs 5 and 10).

are given by

$$\begin{aligned} \omega_{x_3} &= \theta^+ - \beta^+ \sin\zeta \\ \omega_{y_3} &= \zeta^+ \sin\theta - \beta^+ \cos\zeta \cos\theta \\ \omega_{z_3} &= \zeta^+ \cos\theta + \beta^+ \cos\zeta \sin\theta \end{aligned} \tag{1}$$

The rotational transformation between the $x_3y_3z_3$ and xyz triads (figure 2) can be written in terms of direction cosines as

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{2}$$

For the flap-lag-pitch sequence being considered the explicit form of equation 2 is

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} \cos\beta\cos\zeta & \sin\zeta & \cos\zeta\sin\beta \\ -\sin\zeta\cos\beta\cos\theta - \sin\beta\sin\theta & \cos\zeta\cos\theta & \cos\beta\sin\theta - \sin\zeta\sin\beta\cos\theta \\ \sin\zeta\cos\beta\sin\theta - \sin\beta\cos\theta & -\cos\zeta\sin\theta & \sin\zeta\sin\beta\sin\theta + \cos\beta\cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{3}$$

There now remains the task of expressing the rotation angles β, ζ, θ and their derivatives with respect to s_3 in terms of u, v, w, ϕ, and θ_{pt}. The direction cosines ℓ_1, m_1, n_1 of x_3 with respect to xyz can be written using equation [illegible] of reference 5 as

$$\begin{aligned} \ell_1 &= (1+2\varepsilon)^{-1/2} (1+u') = \cos\beta\cos\zeta \\ m_1 &= (1+2\varepsilon)^{-1/2} v' = \sin\zeta \\ n_1 &= (1+2\varepsilon)^{-1/2} w' = \cos\zeta\sin\beta \end{aligned} \tag{4}$$

From equations 4

$$\sin\beta = \frac{v'}{\sqrt{1 + 2\varepsilon - v'^2}} \qquad \cos\beta = \frac{1 + u'}{\sqrt{1 + 2\varepsilon - v'^2}}$$

$$\sin\zeta = \frac{v'}{\sqrt{1 + 2\varepsilon}} \qquad \cos\zeta = \frac{\sqrt{1 + 2\varepsilon - v'^2}}{\sqrt{1 + 2\varepsilon}} \tag{5}$$

where ε is the extensional component of Green's strain tensor and is given by

$$\varepsilon = u' + 1/2\ (u'^2 + v'^2 + w'^2) \tag{6}$$

From the definition of Green's strain tensor, the derivatives with respect to s_3 are related to the derivatives with respect to x according to

$$(\ \ )^+ = (1+2\varepsilon)^{-1/2}\ (\ \ )' \tag{7}$$

The third rotation angle θ specifies the orientation of the y_3z_3 axes with respect to y_2z_2. This additional rotation is due to torsion of the blade, ϕ, about the x_2 axis in the absence of pretwist. Common practice in the rotor blade literature is to combine the pretwist with the elastic torsion. Using this expedient, the third rotation angle θ in the present development is given by

$$\theta = \theta_{pt} + \phi \tag{8}$$

As a result of the deformations, the line element dx becomes an element of arc ds_3 of the deformed elastic axis. In view of this, the expressions given in equations 4 are the direction cosines of the tangent (x_3 axis) to the deformed elastic axis. The x_3,y_3,z_3 coordinate axes are aligned along the tangent, normal, and binormal directions of the deformed elastic axis. These coordinate axes and the trigonometric relations given by equations 5 are shown in figure 4.

Lag-flap-pitch transformation sequence: For this rotational transformation sequence (figure 5) the rotations are imposed as follows:

1. A positive rotation ζ about the z axis resulting in $x_1y_1z_1$
2. A positive rotation β about the negative y_1 axis resulting in $x_2y_2z_2$
3. A positive rotation θ about the x_2 axis resulting in $x_3y_3z_3$

Using figure 5, the bending curvatures ω_{y_3} and ω_{z_3}, and the torsional curvature ω_{x_3}, are obtained by projecting β^+, ζ^+, and θ^+ along the $x_3y_3z_3$ axes and are given by

$$\begin{aligned}
\omega_{x_3} &= \theta^+ + \zeta^+ \sin\beta \\
\omega_{y_3} &= \zeta^+ \cos\beta \sin\theta - \beta^+ \cos\theta \\
\omega_{z_3} &= \zeta^+ \cos\beta \cos\theta + \beta^+ \sin\theta
\end{aligned} \tag{9}$$

For the lag-flap-pitch sequence, the explicit form of equation 2 is

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} \cos\beta\cos\zeta & \sin\zeta\cos\beta & \sin\beta \\ -\cos\zeta\sin\beta\sin\theta - \sin\zeta\cos\theta & \cos\zeta\cos\theta - \sin\zeta\sin\beta\sin\theta & \cos\beta\sin\theta \\ -\cos\zeta\sin\beta\cos\theta + \sin\zeta\sin\theta & -\sin\zeta\sin\beta\cos\theta - \cos\zeta\sin\theta & \cos\beta\cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{10}$$

As before, the direction cosines ℓ_1, m_1, n_1 of x_3 with respect to xyz can be written using reference 5 as

$$
\begin{aligned}
\ell_1 &= (1+2\varepsilon)^{-1/2} (1+u') = \cos\beta \cos\zeta \\
m_1 &= (1+2\varepsilon)^{-1/2} v' = \sin\zeta \cos\beta \\
n_1 &= (1+2\varepsilon)^{-1/2} w' = \sin\beta
\end{aligned}
\tag{11}
$$

From equations 11

$$
\sin\beta = \frac{w'}{\sqrt{1+2\varepsilon}} \qquad \cos\beta = \frac{\sqrt{1+2\varepsilon-w'^2}}{\sqrt{1+2\varepsilon}}
$$

$$
\sin\zeta = \frac{v'}{\sqrt{1+2\varepsilon-w'^2}} \qquad \cos\zeta = \frac{1+u'}{\sqrt{1+2\varepsilon-w'^2}}
\tag{12}
$$

Again, the third rotation angle is taken to be given by equation 8. Similar to the flap-lag-pitch case discussed above, equations 11 give the direction cosines of the tangent (x_3 axis) to the deformed elastic axis. As before, the $x_3y_3z_3$ coordinate axes are aligned along the tangent, normal, and binormal of the deformed elastic axis. These coordinate axes and the trigonometric relations given by equations 12 are shown in figure 5.

The exact nonlinear curvature expressions and rotational transformation matrices given above, which are highly nonlinear, may be approximated to any desired degree by applying the binomial theorem and substituting the trigonometric expansions for β, ζ, and θ. In the following sections, the general expressions will be reduced to four levels of approximation. In each of these levels only terms through second degree in u, v, w, and ϕ will be retained. The transformation matrices to second degree will be given for the three cases of small deformation considered.

<u>Large deformations</u>. - For the particular case of large deformations considered herein the second-degree nonlinear expressions for the curvatures are obtained by substituting equations 5, 6, 7, and 8 into equation 1 for a flap-lag-pitch transformation sequence and equations 6, 7, 8, and 12 into equations 9 for a lag-flap-pitch transformation sequence, expanding the resulting expressions in a binomial series, and retaining terms up to second degree in u, v, w, and ϕ. The resulting curvature expressions assume the form

$$
\begin{aligned}
\omega_{x_3} &= \left(1 - u' - \frac{1}{2} v'^2 - \frac{1}{2} w'^2\right) \theta'_{pt} + \phi'(1-u') - w'' v' \\
\omega_{y_3} &= (-w'' + 2u'w'' + u''w') \cos\theta_{pt} + \phi w'' \sin\theta_{pt} \\
&\quad + (v'' - 2u'v'' - u''v') \sin\theta_{pt} + \phi v'' \cos\theta_{pt} \qquad (13) \\
\omega_{z_3} &= (v''-2u'v'' - u''v')\cos\theta_{pt} - \phi v'' \sin\theta_{pt} \\
&\quad + (w''-2u'w'' - u''w') \sin\theta_{pt} + \phi w'' \cos\theta_{pt}
\end{aligned}
$$

for a flap-lag-pitch sequence and

$$\omega_{x_3} = (1-u' - \frac{1}{2} v'^2 - \frac{1}{2} w'^2)\ \theta'_{pt} + \phi'(1-u') + v''w'$$

$$\omega_{y_3} = (-w''+2u'w'' + u''w')\ \cos\theta_{pt} + \phi\ w''\sin\theta_{pt}$$

$$+ (v'' - 2u'v'' - u''w')\ \sin\theta_{pt} + \phi\ v''\ \cos\ \theta_{pt} \qquad (14)$$

$$\omega_{z_3} = (v''-2u'v'' - u''v')\ \cos\theta_{pt} - \phi\ v''\ \sin\theta_{pt}$$

$$+ (w''-2u'w'' - u''w')\ \sin\theta_{pt} + \phi\ w''\ \cos\theta_{pt}$$

for a lag-flap-pitch sequence.

Small deformations I. - The majority of engineering materials are elastic only for small deformations characterized by elongations and shears which are negligible compared to unity. This implies that the strains are also negligible compared to unity. Since the rotations are arbitrary, the shears can be neglected in comparison with the rotations in determining the direction cosines of the fibers of the beam in the deformed state. This implies that plane sections remain plane (ref. 5, page 47). With the assumptions of this level of approximation the principal axes of the deformed blade differ little from the directions of the normal and binormal to the deformed elastic axis. More detailed considerations on these aspects are given in references 5 and 10. Invoking the assumptions of this level the trigonometric relations of equations 5 and 12, after expanding in a binomial series and retaining terms up to second degree in u, v, and w, can be cast into the form

$$\begin{aligned} \sin\beta &\approx w' & \cos\beta &\approx 1 - \frac{1}{2}(u'^2 + w'^2) \\ \sin\zeta &\approx v' & \cos\zeta &\approx 1 - \frac{1}{2} v'^2 \end{aligned} \qquad (15)$$

and

$$\begin{aligned} \sin\beta &\approx w' & \cos\beta &\approx 1 - \frac{1}{2} w'^2 \\ \sin\zeta &\approx v' & \cos\zeta &\approx 1 - \frac{1}{2}(u'^2 + v'^2) \end{aligned} \qquad (16)$$

It should be observed that the second-degree expressions given in equation 15 do not satisfy the trigonometric identity

$$\sin^2\beta + \cos^2\beta = 1$$

and the expressions in equation 16 do not satisfy the identity

$$\sin^2\zeta + \cos^2\zeta = 1$$

unless u'^2 is negligible compared to unity or, equivalently, u'^2 is negligible compared to both v'^2 and w'^2. This implies that, under the assumption that the elongations and shears are negligible compared to unity, the term u'^2 in the extensional component of Green's strain tensor given by equation 6 must be discarded in beam applications. This fact has not been pointed out in the literature as far as the authors know. It is interesting to note that this requirement is analogous to the inextensibility assumption invoked by Love (ref. 1) while developing the <u>linear</u> (first degree) curvature expressions for a beam. However, inextensibility has not been assumed in the present development as a consequence of this requirement as far as the second-degree <u>nonlinear</u> curvature expressions are concerned. Invoking the assumptions corresponding to the case of small deformations I and imposing the additional requirement that u'^2 is negligible compared to both v'^2 and w'^2, substituting equations 5, 6, 7, and 8 into equations 1 and equations 6, 7, 8, and 12 into equations 9, expanding the trigonometric expressions in a binomial series, and retaining terms up to second degree , the curvature expressions simplify to

$$\begin{aligned}
\omega_{x_3} &= \theta'_{pt} + \phi' - w''v' \\
\omega_{y_3} &= -w''(\cos\theta_{pt} - \phi\sin\theta_{pt}) + v''(\sin\theta_{pt} + \phi\cos\theta_{pt}) \qquad (17) \\
\omega_{z_3} &= v''(\cos\theta_{pt} - \phi\sin\theta_{pt}) + w''(\sin\theta_{pt} + \phi\cos\theta_{pt})
\end{aligned}$$

for a flap-lag-pitch rotational sequence and

$$
\begin{aligned}
\omega_{x_3} &= \theta'_{pt} + \phi' + v''w' \\
\omega_{y_3} &= -w''(\cos\theta_{pt} - \phi \sin\theta_{pt}) + v'' (\sin\theta_{pt} + \phi \cos\theta_{pt}) \qquad (18) \\
\omega_{y_3} &= v''(\cos\theta_{pt} - \phi \sin\theta_{pt}) + w'' (\sin\theta_{pt} + \phi \cos\theta_{pt})
\end{aligned}
$$

for a lag-flap-pitch rotational sequence. This is the level of approximation usually employed for developing the nonlinear aeroelastic equations of motion of a flexible rotor blade.

Small deformations II. - This case is obtained from the case of small deformations I by imposing the additional assumptions that the rotations are negligible compared to unity and that the elongations and shears are much smaller than the rotations. If the first additional assumption is imposed the second-degree terms in equations 17 and 18 can be neglected compared to the first-degree terms and each of the equations reduces to the first-degree (linear) expression

$$
\begin{aligned}
\omega_{x_3} &= \theta'_{pt} + \phi' \\
\omega_{y_3} &= -w''\cos\theta_{pt} + v''\sin\theta_{pt} \qquad (19) \\
\omega_{z_3} &= v''\cos\theta_{pt} + w''\sin\theta_{pt}
\end{aligned}
$$

It should be noted that the assumption that the elongations and shears are much smaller than the rotations has not been explicitly imposed. This assumption is required, however, in order to insure that the principal axis directions of a cross-section of the deformed blade still differ little from the directions of the normal and binormal of the deformed elastic axis. It should also be pointed out that this is the level of approximation usually employed in elastic stability (buckling) problems.

Classical linear theory. - To arrive at this case from the case of small deformations II the additional assumption is made that the terms of second degree in the angles of rotation are negligible compared to the corresponding strain terms. Herein, the terminology "classical linear theory" means that the geometric relations between strains and displacements are linear. Since there are no terms involving squares of rotations in equations 19 the curvatures for the classical linear theory are again given by equations 19. It should be emphasized, however, that although the curvature expressions corresponding to the last two levels of approximation are the same, the strain components are different (see ref. 5).

Direction Cosine Approach

For completeness it is appropriate to indicate how one would extend the approach used by Love (ref. 1) for developing the linear curvature expressions to the general case of large deformations. The rotational transformation between the $x_3y_3z_3$ and xyz triads has already been given in equation 2. Following the procedure in Love, the curvatures ω_{x_3}, ω_{y_3}, ω_{z_3} can be written in terms of the direction cosines of equation 2 as

$$
\begin{aligned}
\omega_{x_3} &= (1+2\varepsilon)^{-1/2} \left(\ell_3\ell_2' + m_3m_2' + n_3n_2'\right) \\
\omega_{y_3} &= (1+2\varepsilon)^{-1/2} \left(\ell_1\ell_3' + m_1m_3' + n_1n_3'\right) \\
\omega_{z_3} &= (1+2\varepsilon)^{-1/2} \left(\ell_2\ell_1' + m_2m_1' + n_2n_1'\right)
\end{aligned}
\qquad (20)
$$

The factor $(1 + 2\varepsilon)^{-1/2}$ in equations 20 accounts for the fact that the derivatives are taken with respect to the undeformed coordinate x.

The direction cosines in equations 20 depend on the order in which the rotational transformation sequence between xyz and $x_3y_3z_3$ is imposed. For the two transformation sequences considered above, the explicit forms of equations 2 are given by equation 3 for a flap-lag-pitch sequence and equation 10 for a lag-flap-pitch sequence. The angles β, ζ, and θ which appear in equations 3

and 10 are defined in equations 5, 6, 8, and 12 and thus the direction cosines are known in terms of u, v, w, ϕ, and θ_{pt}. The curvature expressions for the general case of large deformations are obtained by substituting the direction cosines appropriate to both the flap-lag-pitch and lag-flap-pitch transformation sequences into equation 20. As remarked earlier these general expressions are highly nonlinear and may be approximated to any desired degree by applying the binomial theorem. In this section the general expressions will again be reduced to four levels of approximation and only second-degree terms in u, v, w, and ϕ will be retained.

For the particular case of large deformations considered herein the second-degree nonlinear expressions for the curvatures are obtained by using equations 3, 5, 6, and 8 in equation 20 for the flap-lag-pitch sequence and using equations 6, 8, 10 and 12 in equation 20 for the lag-flap-pitch sequence, expanding the resulting expressions, and retaining terms up to second degree in u, v, w, and ϕ. The expressions thus obtained are identical with the second-degree curvature expressions given previously in equations 13 and 14.

If the assumptions corresponding to the case of small deformations I are imposed, the second-degree expressions for the direction cosines associated with a flap-lag-pitch rotational transformation sequence, from equations 2, 3, 5, 6, and 8, can be shown to be

$$
\begin{aligned}
\ell_1 &= 1 - \frac{1}{2}(u'^2 + v'^2 + w'^2) \\
\ell_2 &= -v'(\cos\theta_{pt} - \phi\sin\theta_{pt}) - w'(\sin\theta_{pt} + \phi\cos\theta_{pt}) \\
\ell_3 &= v'(\sin\theta_{pt} + \phi\cos\theta_{pt}) - w'(\cos\theta_{pt} - \phi\sin\theta_{pt}) \\
m_1 &= v' \\
m_2 &= \left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta_{pt} - \phi\sin\theta_{pt} \qquad (21) \\
m_3 &= -\left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta_{pt} - \phi\cos\theta_{pt}
\end{aligned}
$$

$$n_1 = w'$$

$$n_2 = (\phi - v'w')\cos\theta_{pt} + \left(1 - \frac{u'^2}{2} - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta_{pt}$$

$$n_3 = (v'w' - \phi)\sin\theta_{pt} + \left(1 - \frac{u'^2}{2} - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta_{pt}$$

and those associated with a lag-flap-pitch rotational transformation sequence, from equations 2, 6, 8, 10, and 12, to be

$$\ell_1 = 1 - \frac{1}{2}(u'^2 + v'^2 + w'^2)$$

$$\ell_2 = v'(\cos\theta_{pt} - \phi\sin\theta_{pt}) - w'(\sin\theta_{pt} + \phi\cos\theta_{pt})$$

$$\ell_3 = v'(\sin\theta_{pt} + \phi\cos\theta_{pt}) - w'(\cos\theta_{pt} - \phi\sin\theta_{pt})$$

$$m_1 = v'$$

$$m_2 = -(v'w' + \phi)\sin\theta_{pt} + \left(1 - \frac{u'^2}{2} - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta_{pt} \qquad (22)$$

$$m_3 = (-v'w' + \phi)\cos\theta_{pt} + \left(1 - \frac{u'^2}{2} - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta_{pt}$$

$$n_1 = w'$$

$$n_2 = \left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta_{pt} + \phi\cos\theta_{pt}$$

$$n_3 = \left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta_{pt} - \phi\sin\theta_{pt}$$

It should be observed that the direction cosines given in equations 21 and 22 do not satisfy the orthogonality property

$$\ell_1^2 + m_1^2 + n_1^2 = 1$$

unless u'^2 is assumed negligible compared to unity or, equivalently, u'^2 is assumed negligible compared to both v'^2 and w'^2. This is the same conclusion reached earlier on the basis of trigonometric identities. Again, these results reaffirm that if one assumes that the elongations and shears are negligible compared to unity the additional requirement that u'^2 is negligible compared to both v'^2 and w'^2 must also be imposed. Substituting equations 21 and 22 into equations 20 and invoking the above additional assumption the second-degree nonlinear curvature expressions given in equations 17 and 18 are again obtained. The corresponding rotational transformation matrices between the deformed and undeformed blade-fixed coordinates are given by

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}\left(v'^2 + w'^2\right) & v' & w' \\ -v'\left(\cos\theta_{pt} - \phi\sin\theta_{pt}\right) & \left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta_{pt} & \left(\phi - v'w'\right)\cos\theta_{pt} \\ -w'\left(\sin\theta_{pt} + \phi\cos\theta_{pt}\right) & -\phi\sin\theta_{pt} & +\left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta_{pt} \\ v'\left(\sin\theta_{pt} + \phi\cos\theta_{pt}\right) & -\left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta_{pt} & \left(v'w' - \phi\right)\sin\theta_{pt} \\ -w'\left(\cos\theta_{pt} - \phi\sin\theta_{pt}\right) & -\phi\cos\theta_{pt} & +\left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta_{pt} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{23}$$

for a flap-lag-pitch rotational transformation sequence and

$$\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} 1 - \frac{1}{2}\left(v'^2 + w'^2\right) & v' & w' \\ -v'\left(\cos\theta_{pt} - \phi\sin\theta_{pt}\right) & -\left(\phi + v'w'\right)\sin\theta_{pt} & \left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta_{pt} \\ -w'\left(\sin\theta_{pt} + \phi\cos\theta_{pt}\right) & +\left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta_{pt} & +\phi\cos\theta_{pt} \\ v'\left(\sin\theta_{pt} + \phi\cos\theta_{pt}\right) & -\left(\phi + v'w'\right)\cos\theta_{pt} & \left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta_{pt} \\ -w'\left(\cos\theta_{pt} - \phi\sin\theta_{pt}\right) & -\left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta_{pt} & -\phi\sin\theta_{pt} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{24}$$

for a lag-flap-pitch rotational transformation sequence. The case of small deformations II is obtained by imposing the additional restrictions that the rotations are negligible compared to unity and that the elongations and shears are much smaller than the rotations. In this case, the second-degree terms in equations 23 and 24 can be neglected compared to the first-degree terms and reduce to the single relation

$$
\begin{Bmatrix} x_3 \\ y_3 \\ z_3 \end{Bmatrix} = \begin{bmatrix} 1 & v' & w' \\ -v' \cos\theta_{pt} - w' \sin\theta_{pt} & \cos\theta_{pt} - \phi\sin\theta_{pt} & \sin\theta_{pt} + \phi\cos\theta_{pt} \\ v' \sin\theta_{pt} - w' \cos\theta_{pt} & -\sin\theta_{pt} - \phi\cos\theta_{pt} & \cos\theta_{pt} - \phi\sin\theta_{pt} \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \qquad (25)
$$

The transformation given in equation 25 is linear and is thus independent of the order in which the rotations are imposed. Substituting the direction cosines given in equation 25 into equations 20 the same first-degree (linear) curvatures given by equations 19 are obtained. Formally, some second degree terms also arise while making this substitution. However, these second-degree terms have no meaning and must be discarded since the direction cosines from the linear transformation matrix of equation 25 do not satisfy the direction cosine orthogonality relations to second degree. The need to discard these

nonlinear terms is also dictated by the assumption that the rotations are negligible compared to unity.

Again, the curvatures corresponding to classical linear theory are the same as those corresponding to the case of small deformations II.

It was indicated earlier that the second-degree curvature expressions in references 6 and 7 were obtained by solving a differential equation for the transformation matrix which relates the blade-fixed coordinates of the deformed and undeformed blade. If one employs that procedure using the transformation matrices given in equations 23 and 24 and solves for the unknowns, ω_{x_3}, ω_{y_3}, and ω_{z_3} one will obtain the same curvature expressions given in equations 17 and 18.

The second-degree nonlinear curvature expressions for a twisted rotor blade or beam have been established above in two ways for four levels of approximation. The same expressions can also be developed if foreshortening due to bending (ref. 8) is explicitly included in the axial deformation. In the next section the curvature expressions and transformation matrices developed herein will be compared with corresponding results existing in the literature.

COMPARISONS AND DISCUSSION

The only general development in the literature for the curvature expressions, as far as the authors know, is that of Hunter (ref. 2). Since reference 2 considers initial (built-in) curvatures before imposing the elastic deformations and the present development combines pretwist with elastic twist and considers them as the last rotation, a direct comparison of the present results with those of reference 2 is not possible. A comparison can be made, however, if the initial curvatures in reference 2 and the pretwist in the present development are set to zero. Under these conditions the present second-degree nonlinear expressions given by equations 13 for a flap-lag-pitch sequence and equations 14 for a lag-flap-pitch sequence agree with the corresponding expressions obtained from reference 2 after making the necessary notational changes.

Several investigators have developed second-degree nonlinear curvature expressions assuming small deformations. Hodges (ref. 4) derived these expressions for zero pretwist. Though not stated explicitly, the level of approximation employed is equivalent to the case of small deformation I herein. However, reference 4 discarded u'^2 through an a priori ordering scheme rather than through a rigorous argument. In addition to these assumptions, reference 4 used a partially nonlinear displacement field in u, v, w, and ϕ. The use of this displacement field means that the rotational transformation matrix between the deformed and undeformed blade-fixed coordinates was partially linearized. If one is developing a linear set of curvature expressions, a linear displacement field in u, v, w, and ϕ and a linear resultant rotational transformation matrix are sufficient and the order in which the rotations are imposed is not important. However, if one is developing the second-degree nonlinear curvature expressions care must be taken to insure retention of all second-degree terms in the displacement field and in the resultant transformation matrix and the order in which the rotations are imposed is important. Since reference 4 did not address the rotational transformation sequence explicitly and has employed an incomplete displacement field, a meaningful comparison with the present results can not be made. However, it is interesting to note that the second-degree curvature expressions of reference 4 agree with those associated with the flap-lag-pitch sequence given herein after making the necessary notational changes. In view of the use of a partially nonlinear displacement field, this agreement must be regarded as fortuitous.

References 6 and 7 have also addressed the problem of deriving nonlinear curvature expressions for a hingeless rotor blade. No indication is given in reference 6 as to the level of approximation employed. The assumptions made in reference 7 are equivalent to those herein corresponding to the case of small deformations I. The term u'^2 was discarded in reference 7 on the basis of an ordering scheme. Both these references used a lag-flap-pitch rotational transformation sequence between the deformed and the undeformed blade-fixed coordinates. Reference 6 a priori identified the torsional curvature as ϕ', the pretwist being zero. Reference 7 also a priori identified the torsional curvature as being equal to $(\theta_{pt} + \phi)'$. The definition of ϕ herein is the

same as that given in references 6 and 7. The results of the cited references disagree with the corresponding torsional curvature for the lag-flap-pitch sequence derived herein and given by $\omega_{x_3} = \theta'_{pt} + \phi' + v''w'$. The bending curvatures of reference 7 are, however, in agreement with the corresponding ones herein. As a consequence of these a priori identifications the lag-flap-pitch rotational transformation matrix of references 6 and 7 do not agree with the corresponding one given by equation 24 herein.

Second-degree nonlinear curvature expressions have been developed in reference 3 for application to dynamic analyses of tubes conveying fluids. The level of approximation employed in reference 3 is equivalent to the case of small deformations I herein. Since the rotational transformation sequence was not specified a comparison was made with both sets of the present results after setting the initial curvatures and twist of reference 3 and the pretwist in the present results to zero. This comparison showed that the torsional curvature ω_{x_3} and the bending curvature ω_{z_3} of reference 3 expressed in the present notation agree with the corresponding results herein. The bending curvature ω_{y_3} is not in agreement with the present results. The detailed development leading to the curvature expressions given in reference 3 were presented in reference 11. An examination of reference 11 reveals that the disagreement with the present results is a consequence of the linearization of the resultant transformation matrix between the deformed and undeformed blade-fixed coordinates while developing the nonlinear curvature expressions. This linearization is not consistent with the assumptions of small deformations I although it is justified under the assumptions of small deformations II. However, as already stated the nonlinear terms in the curvature expressions under the assumptions of small deformations II have no meaning and must be discarded.

Under the assumptions of classical linear theory the curvatures are linear and are again given by equations 19. The linear curvatures can be obtained from the nonlinear curvatures of the case of small deformations I by discarding the squares and products of the elastic variables u, v, w, and ϕ and their derivatives. The linear curvature expressions obtained in this manner in references 2, 3, 4, 7, and 8 are in agreement with the corresponding ones herein.

During the present development an interesting observation has been made regarding the u'^2 term in the extensional component of Green's strain tensor (eq. 6). These observations have a direct bearing on an existing controversial issue regarding whether the uncoupled extensional frequency of a rotating beam increases or decreases with increasing rotational speed. Reference 12, while studying only the extensional vibrations of a rotating beam, used the extensional component of Green's strain tensor in the form

$$\varepsilon = u' + \frac{1}{2} u'^2 \qquad (26)$$

but was able to discard the terms associated with u'^2 in the development by invoking the small strain assumption. This can be formally shown by rearranging equation 26 into the form

$$1 + 2\varepsilon = (1 + u')^2 \qquad (27)$$

If the strain ε in equation 27 is assumed negligible compared to unity then u' must also be assumed negligible compared to unity. Based on this assumption u'^2 must be discarded in equation 26. With this assumption reference 12 concluded that the extensional frequency of a rotating beam decreases with increasing rotational speed. It should be remarked that in the more general case including flapwise and edgewise bending the conclusion that u'^2 must be discarded can not be established on the basis of the above reasoning. However, in the present development this result was established in the general case on the basis of considerations related to satisfying certain trigonometric and direction cosine identities. More recently, reference 13 contradicted the result of reference 12 and concluded that the extensional frequency of a rotating beam increases with increasing rotational speed. In this reference the coupled flapwise and extensional equations of motion of a rotating beam were derived using Green's strain tensor including the u'^2 term under the small strain assumption. Since the extensional equation of motion was obtained as a special case of the coupled equations, reference 13 was apparently unable to recognize

that u'^2 must be neglected compared to w'^2, as identified herein by the trigonometric and direction cosine identities. These considerations reveal that the frequency increases with increasing rotational speed if u'^2 is retained and decreases with increasing rotational speed if u'^2 is discarded. On the basis of the observations made in the present development, under the assumption of small strains, the term u'^2 must be discarded in the extensional component of Green's strain tensor in beam applications. Thus, the present authors are in agreement with the result of reference 12 that, under the assumption of small strains, the uncoupled extensional frequency of a rotating beam decreases with increasing rotational speed.

CONCLUDING REMARKS

A development was presented for the nonlinear expressions for the bending and torsional curvatures of a twisted helicopter rotor blade or beam undergoing combined flapwise bending, chordwise bending, torsion, and extension. The general nonlinear expressions developed are valid for large deformations. These general expressions were systematically reduced to four levels of approximation and in each of these levels the second-degree nonlinear expressions were given. In specializing to the particular case of small deformations in which the elongations and shears (and hence strains) are assumed negligible compared to unity with no restrictions on the rotations, trigonometric and direction cosine anomalies were identified. To remove these anomalies, it was shown that one must additionally impose the requirement that the square of the first derivative of the extensional deformation on the elastic axis be negligible compared to unity and the squares of the bending slopes on the elastic axis. This fact has not been pointed out in the literature as far as the authors know. Using this fact, a controversy existing in the literature regarding whether the uncoupled extensional frequency of a rotating beam increases or decreases with increasing rotational speed has been resolved. Furthermore, the second-degree nonlinear curvature expressions and the rotational transformation matrices between the deformed and undeformed blade-fixed coordinates were compared with corresponding expressions in the literature wherever

possible. These comparisons indicated several discrepancies with the present results in the nonlinear terms. The reasons for these discrepancies were explained.

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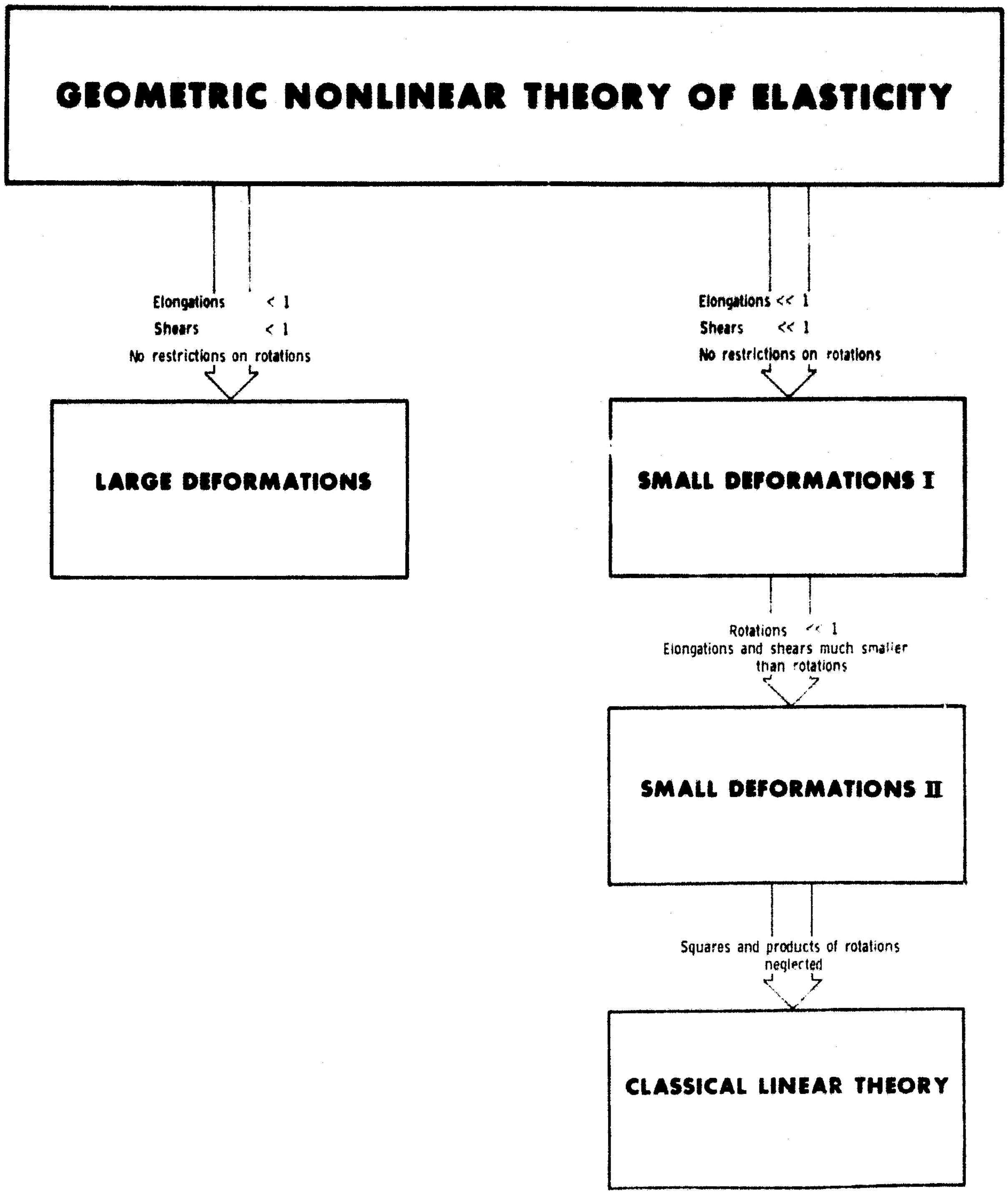

Figure 1.- Levels of approximation within the geometric nonlinear theory of elasticity which are addressed.

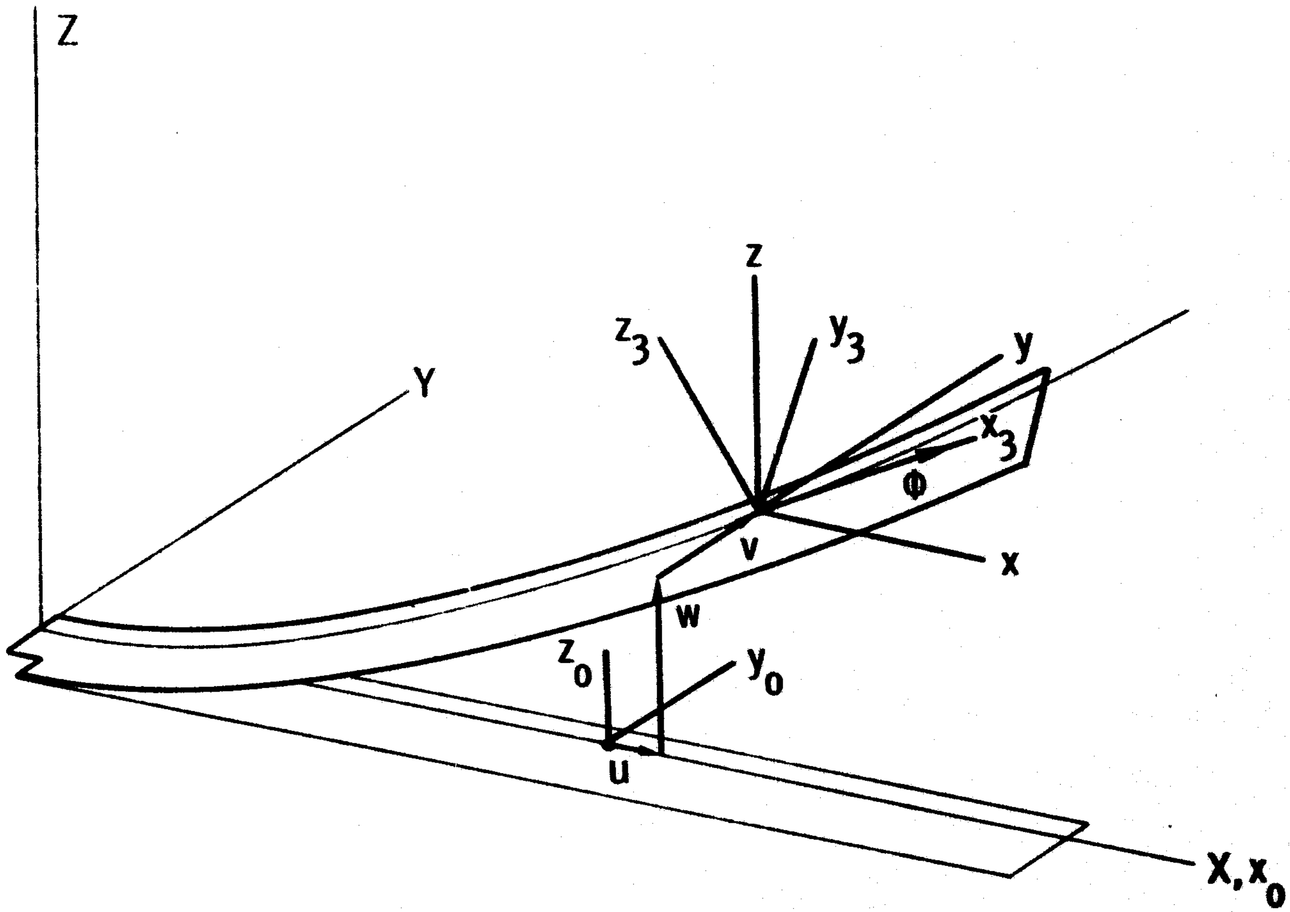

Figure 2.- Schematic representation of undeformed and deformed blade for flap-lag-pitch rotational transformation sequence.

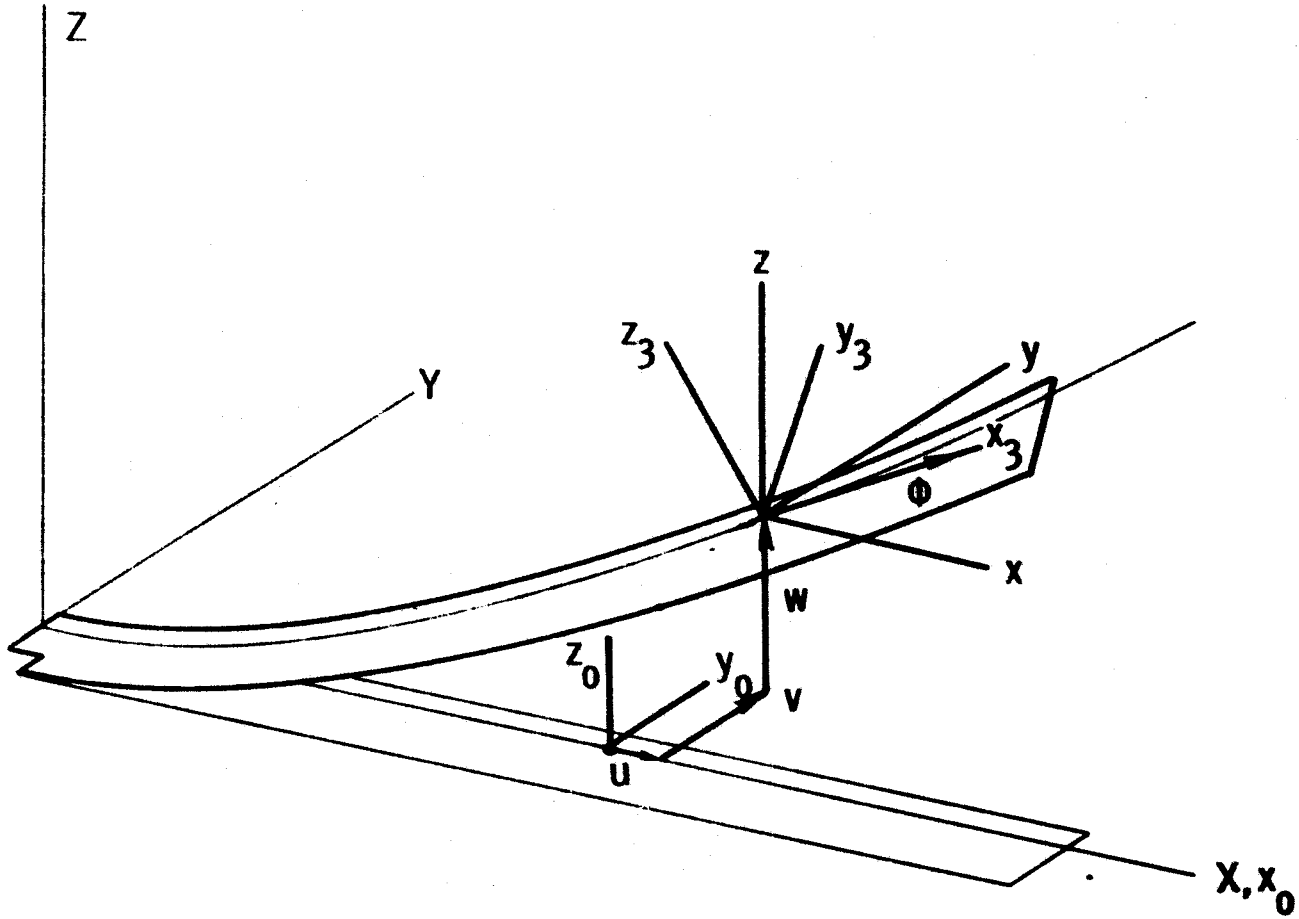

Figure 3.- Schematic representation of undeformed and deformed blade for lag-flap-pitch rotational transformation sequence.

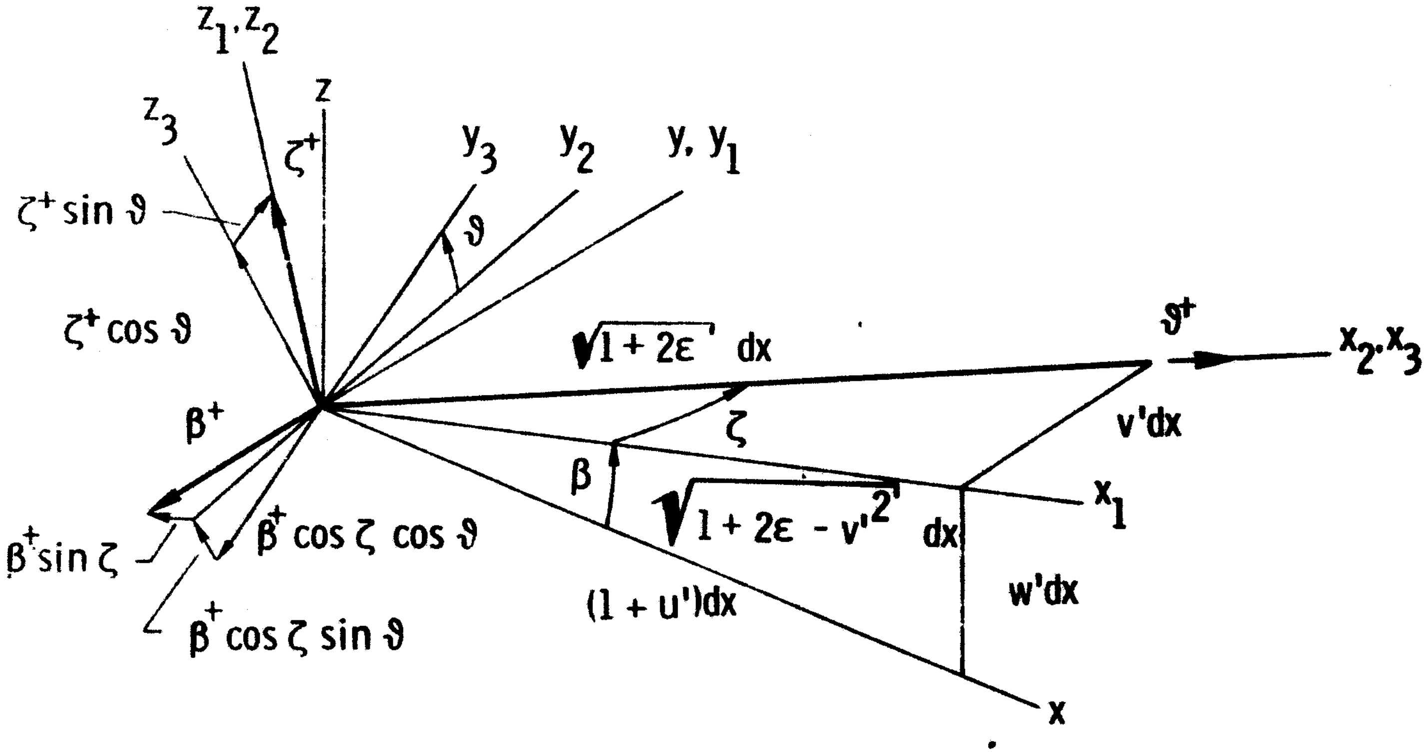

Figure 4.- Rotations between blade-fixed coordinates before and after deformation — flap-lag-pitch rotational transformation sequence.

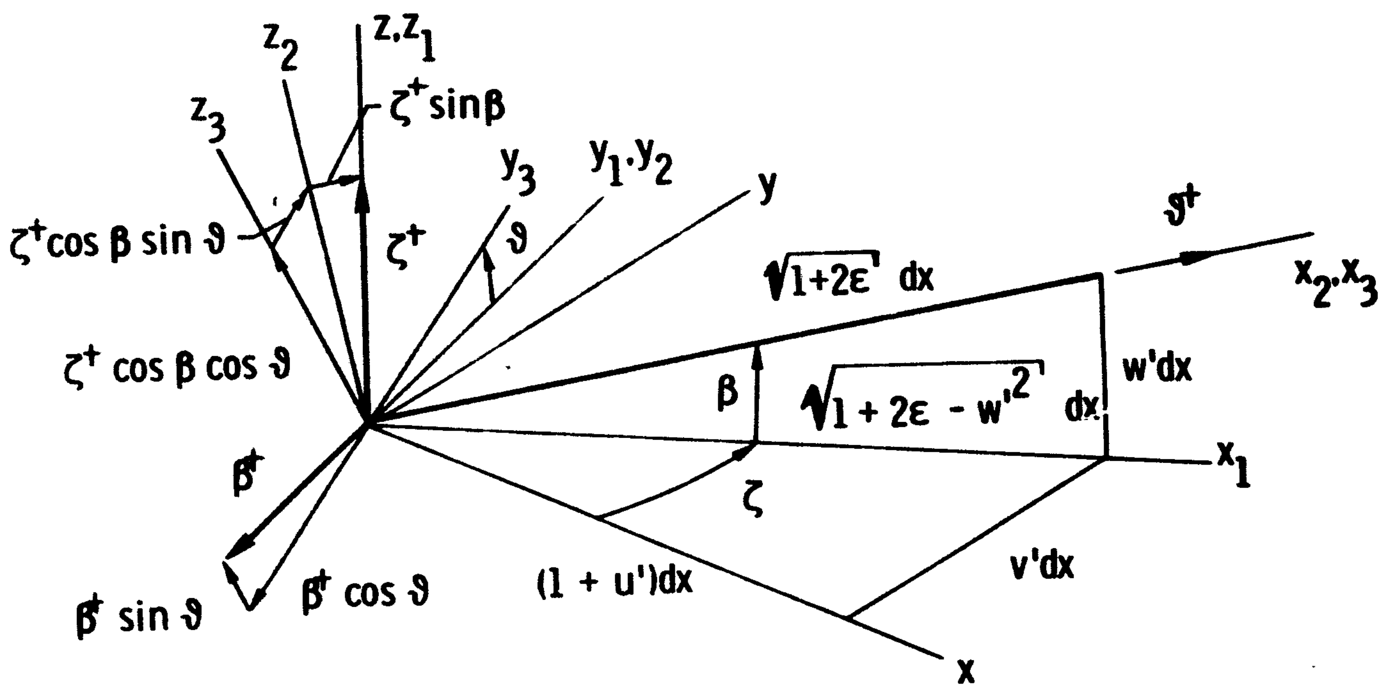

Figure 5.- Rotations between blade-fixed coordinates before and after deformation — lag-flap-pitch rotational transformation sequence.